

ON S. GRIVAUX' EXAMPLE OF A HYPERCYCLIC RANK ONE PERTURBATION OF A UNITARY OPERATOR.

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ABSTRACT. Recently, Sophie Grivaux showed that there exists a rank one perturbation of a unitary operator in a Hilbert space which is hypercyclic. We give a similar construction using a functional model for rank one perturbations of singular unitary operators.

1. INTRODUCTION

A continuous linear operator T in a Fréchet space F is said to be *hypercyclic* if there exists a vector $f \in F$ such that its orbit $\{T^n f\}_{n=0}^\infty$ is dense in F . In this case the vector f is said to be *hypercyclic for T* . First examples of hypercyclic operator go back to G.D. Birkhoff, G.R. McLane, S. Rolewicz, while a systematic study of hypercyclicity phenomenon started in 1980-s with the thesis by C. Kitai and works by R.M. Gethner, G. Godefroy and J.H. Shapiro [5, 6]. We refer to the recent monographs [2, 9] for a detailed account of the theory.

Clearly, the identity operator is one of the most "nonhypercyclic" operators. However, already in 1991 K.C. Chan and J.H. Shapiro [3] showed that there exists hypercyclic operators in a Hilbert space of the form $I + K$, where the compact operator K may belong to any Schatten class. It is clear that $I + R$ can not be hypercyclic when R is a finite rank operator. Still, if we replace I by a unitary operator, hypercyclicity is possible. In 2010 S. Shkarin [14] produced an example of a unitary operator U such that $U + R$ is hypercyclic for some rank two operator R . Shkarin asked whether R can be taken to be of rank one. A positive answer was given by S. Grivaux [8]:

Theorem 1.1 (S. Grivaux, [8]). *There exists a unitary operator U in the space ℓ^2 and a rank one operator R such that $U + R$ is hypercyclic.*

The proof of this theorem is based on an ingenious elementary construction involving a certain convergent inductive process as well as on the following sufficient condition for hypercyclicity also obtained by Grivaux [7]. This result says that the operator is hypercyclic if there is a certain "continuous" family of eigenvectors with unimodular eigenvalues.

Theorem 1.2 (S. Grivaux, [7]). *Let X be a complex separable infinite-dimensional Banach space, and let T be a bounded operator on X . Suppose that there exists a sequence $\{u_n\}_{n \geq 1}$ of vectors in X having the following properties:*

- (i) u_n is an eigenvector of T associated to an eigenvalue λ_n of T , with $|\lambda_n| = 1$ and λ_n are all distinct;
- (ii) $\text{span}\{u_n : n \geq 1\}$ is dense in X ;

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(iii) for any $n \geq 1$ and any $\varepsilon > 0$, there exists $m \neq n$ such that $\|u_n - u_m\| < \varepsilon$.
Then T is hypercyclic and even frequently hypercyclic.

The aim of the present paper is to give a proof of Theorem 1.1 by function theory methods. Our approach is based on a functional model for rank one perturbations of singular unitary operators. This model essentially goes back to a paper by V.V. Kapustin [10]. In the form that we will use, this model appeared in [1, Theorem 0.6] in the context of rank one perturbations of selfadjoint operators. This model translates any rank one perturbation of a unitary operator to some concrete operator in a space of analytic functions in the unit disk which is known as a star-invariant or model subspace (due to its role in yet another model – that of B. Sz.-Nagy and C. Foias).

Let H^2 denote the standard Hardy space in the unit disk, and let θ be an inner function in the disk. The *model* (or *star-invariant*) *subspace* K_θ of H^2 is then defined as

$$K_\theta = H^2 \ominus \theta H^2.$$

According to the famous Beurling theorem, any closed subspace of H^2 invariant with respect to the backward shift in H^2 is of the form K_θ . These subspaces play a distinguished role in operator theory (see, e.g., [11, 12]) and in operator-related complex analysis.

The details on the functional model for rank one perturbations will be given in the next section. For the moment, let us mention only that in this model the family of the eigenvectors of a rank one perturbation have a very transparent analytic meaning: they are either the families of reproducing kernels of K_θ or their biorthogonals.

Now we state our main result which says that there exist model spaces with a certain continuous family of vectors analogous to the properties of the vectors in Theorem 1.2. In view of the functional model (see Theorem 2.1) and Theorem 1.2 this immediately implies that there exists a unitary operator which has a hypercyclic rank one perturbation.

Theorem 1.3. *There exists an inner function θ in the disk such that $\theta(0) \neq 0$ and θ is analytically continuable across some nonempty open subarc of \mathbb{T} , a function $\varphi \in H^2 \setminus K_\theta$ and a sequence $\lambda_n \in \mathbb{T}$ such that the functions*

$$(1) \quad f_n(z) = \frac{\varphi(z)}{z - \lambda_n} \in K_\theta,$$

the family $\{f_n\}$ is complete in K_θ and, for any $n \geq 1$ and any $\varepsilon > 0$, there exists $m \neq n$ such that $\|f_n - f_m\| < \varepsilon$.

We do not by any means claim that our proof is essentially shorter than the original proof by Grivaux. However, we believe that the application of the model clarifies the construction of the eigenvectors, since in the model space they have a special analytic structure: they are necessarily of the form (1) for some $\varphi \in H^2$.

As the original proof from [8], our construction is also inductive. However, the parameters are chosen in a different way. In particular, the eigenvalues of the operator $U + R$ will be chosen to be zeros of some Herglotz function (interlacing with the spectrum of U). The properties of Herglotz functions will play an important role in the construction.

2. PRELIMINARIES ON THE FUNCTIONAL MODEL FOR RANK-ONE PERTURBATIONS OF UNITARY OPERATORS

2.1. Inner functions and Clark measures. Recall that a function θ is said to be *inner* if it is analytic and bounded in the unit disk \mathbb{D} and its nontangential boundary values satisfy $|\theta| = 1$ a.e. with respect to the normalized Lebesgue measure m on the unit circle \mathbb{T} .

Let $H^2 = H^2(\mathbb{D})$ denote the *Hardy space* of the unit disk \mathbb{D} , equipped with the standard norm $\|\cdot\|_2 = \|\cdot\|_{L^2(m)}$. With each inner function θ we associate the model subspace $K_\theta = H^2 \ominus \theta H^2$.

The *reproducing kernel* for K_θ corresponding to a point $\lambda \in \mathbb{D}$ is given by

$$k_\lambda(z) = \frac{1 - \overline{\theta(\lambda)}\theta(z)}{1 - \overline{\lambda}z}.$$

Since functions in K_θ have more analyticity than general H^2 functions, there may exist reproducing kernels at boundary points. In particular, one can consider k_λ , $\lambda \in I$, if θ (and, hence, any function in K_θ) have an analytic continuation across the arc I . More generally, by the results of Ahern and Clark we have $k_\lambda \in K_\theta$ for $\lambda \in \mathbb{T}$ if and only if $|\theta'(\lambda)| < \infty$, the modulus of the angular derivative is finite.

Now we turn to Clark's construction of orthogonal bases of reproducing kernels [4]. For each $\alpha \in \mathbb{T}$, the function $\frac{\alpha + \theta}{\alpha - \theta}$ has positive real part in \mathbb{D} , and so there exists a finite (singular) positive measure μ^α on \mathbb{T} such that

$$\operatorname{Re} \frac{\alpha + \theta(z)}{\alpha - \theta(z)} = \frac{1}{\pi} \int_{\mathbb{T}} \frac{1 - |z|^2}{|\tau - z|^2} d\mu^\alpha(\tau), \quad z \in \mathbb{D}.$$

Clark's theorem states that if, for some α , μ^α is purely atomic, i.e., if $\mu^\alpha = \sum_n \mu_n \delta_{\tau_n}$, $\tau_n \in \mathbb{T}$, then $k_{\tau_n} \in K_\theta$ and the system $\{k_{\tau_n}\}$ is an orthogonal basis in K_θ . Note also that $\|k_{\tau_n}\|_2^2 = |\theta'(\tau_n)| = 2\mu_n^{-1}$.

Let $\mu = \mu^1$ be the Clark measure corresponding to $\alpha = 1$. For $c \in L^2(\mu)$, put

$$(2) \quad (Vc)(z) = (1 - \theta(z)) \int_{\mathbb{T}} \frac{c(\tau)d\mu(\tau)}{1 - \bar{\tau}z}, \quad z \in \mathbb{D}.$$

As Clark [4] has shown, V is a unitary operator from $L^2(\mu)$ onto K_θ . Moreover, the nontangential boundary values of the function Vc exist and coincide with c μ -a.e. [13]. In particular, if $\{k_{\tau_n}\}$ is an orthogonal basis of reproducing kernels in K_θ , then any $f \in K_\theta$ is of the form $f(z) = (1 - \theta(z)) \sum_n \frac{c_n \mu_n}{1 - \bar{\tau}_n z}$, where $\sum_n |c_n|^2 \mu_n < \infty$.

2.2. The functional model. Now we give the details of the functional model for rank one perturbations of unitary operators. Let θ be an inner function in the disk such that $\theta(0) \neq 0$. For a function $f \in K_\theta$, the function zf is not necessarily in K_θ , but it is easily seen that $zf = \gamma + h$, where $\gamma \in \mathbb{C}$ is a constant and $h \in K_\theta$, and such decomposition is unique (moreover, γ is a continuous functional of f). Now let $\varphi \in H^2$ be a function such that

$$(3) \quad \varphi \notin K_\theta, \quad \frac{\varphi(z) - \varphi(0)}{z} \in K_\theta.$$

Then we may define the operator $T = T_{\theta, \varphi}$ on K_θ by the formula

$$(4) \quad Tf := zf - \gamma_f \varphi,$$

where γ_f is the unique complex number such that $zf - \gamma_f \varphi \in K_\theta$.

We are ready to present the functional model of rank one perturbations which is analogous to [1, Theorem 0.6] (though the main ideas go back to [10]). Recall that a unitary operator is said to be *singular* if its spectral measure is singular with respect to the Lebesgue measure on \mathbb{T} . We also assume below that U is cyclic, and so, up to a unitary equivalence, it is the operator of multiplication by z in some space $L^2(\nu)$, where ν is a finite Borel measure on \mathbb{T} . By $\sigma(U)$ we denote the spectrum of U . Finally, we denote by $\rho(\theta)$ the *boundary spectrum* of an inner function θ , that is, the complement of the union of all open arcs I such that θ admits an analytic continuation across I .

Theorem 2.1 (functional model). *Let U be a cyclic singular unitary operator such that $\sigma(U) \neq \mathbb{T}$. Then for any rank one perturbation $U + R$ of U there exist an inner function θ in the disk such that $\theta(0) \neq 0$ and $\rho(\theta) \neq \mathbb{T}$, and a function $\varphi \in H^2$ satisfying (3) such that $U + R$ is unitary equivalent to the operator T defined by (4).*

Conversely, any inner function θ such that $\theta(0) \neq 0$ and $\rho(\theta) \neq \mathbb{T}$, and any function $\varphi \in H^2$ satisfying (3) correspond to some rank one perturbation $U + R$ of a cyclic singular unitary operator U with $\sigma(U) \neq \mathbb{T}$.

It is obvious from the definition of the operator T that if $\lambda \in \mathbb{C}$ is an eigenvalue of T , then the corresponding eigenvector is given by $\frac{\varphi(z)}{z-\lambda}$. Now, combining Theorems 2.1 and 1.2 we see that the existence of a rank one perturbation follows from Theorem 1.3 (since the functions f_n are eigenvectors of some rank one perturbations).

2.3. Model spaces in the upper half-plane. It will be more convenient to work with model spaces in the half-plane \mathbb{C}_+ rather than in the disk. We can translate our problem to the half-plane setting since the map $\tilde{f}(z) = \frac{1}{z+i} \cdot f\left(\frac{z-i}{z+i}\right)$ maps $H^2(\mathbb{D})$ to $H^2(\mathbb{C}_+)$ and the model space K_θ in the disk to the model space $K_{\tilde{\theta}} = H^2(\mathbb{C}_+) \ominus \tilde{\theta}H^2(\mathbb{C}_+)$ in \mathbb{C}_+ , where $\tilde{\theta}(z) = \theta\left(\frac{z-i}{z+i}\right)$.

The definition and the properties of the Clark measures for the model spaces in the upper half-plane are analogous to those in the disk. Indeed, if θ is an inner function in \mathbb{C}_+ , then the function $\frac{\alpha+\theta}{\alpha-\theta}$ has positive real part in \mathbb{C}_+ , the measure μ on \mathbb{R} from its Herglotz representation is said to be a Clark measure for K_θ , and, again, the embedding of K_θ into $L^2(\mu)$ is a unitary operator (with one possible exception where a linear term appears in the Herglotz representation; we will exclude this case in our construction).

In particular, if the Clark measure corresponding to $\alpha = 1$ is purely atomic, $\mu = \sum_n \mu_n \delta_{t_n}$, then any function $f \in K_\theta$ is of the form

$$(5) \quad f(z) = (1 - \theta(z)) \sum_n \frac{c_n \mu_n}{z - t_n}, \quad \sum_n |c_n|^2 \mu_n < \infty,$$

and

$$(6) \quad \|f\|_{L^2(\mathbb{R})}^2 = 4\pi \sum_n |c_n|^2 \mu_n = \pi \sum_n |f(t_n)|^2 \mu_n.$$

We will often use this formula for the norm in what follows.

3. PROOF OF THEOREM 1.3

3.1. Plan of the solution. By the above functional model, the problem is reduced to the construction of functions θ and φ as in Theorem 1.3. As explained in Subsection 2.3, we can work on the real line and in the upper half-plane. Thus, in what follows $\|\cdot\|_2$ stands for the usual L^2 -norm on \mathbb{R} .

We will construct:

- a countable set $T = \{t_n\}_{n=1}^\infty$ on some interval (say, in $[0, 1]$);
- a finite measure $\mu = \sum_{n=1}^\infty \mu_n \delta_{t_n}$;
- an inner function θ defined by

$$(7) \quad \frac{1 + \theta(z)}{1 - \theta(z)} := i \sum_{n=1}^\infty \frac{\mu_n}{z - t_n};$$

- a function φ of the form

$$(8) \quad \varphi(z) := (1 - \theta(z)) \left[\sum_{n=1}^\infty \frac{c_n \mu_n}{t_n - z} + 1 \right], \quad \sum_{n=1}^\infty |c_n|^2 \mu_n < \infty,$$

such that for some sequence $\{\lambda_j\} \subset [0, 1]$, we have $\frac{\varphi}{z - \lambda_j} \in L^2(\mathbb{R})$ and, for any $j \geq 1$ and $\varepsilon > 0$ there exists $k \neq j$ such that

$$(9) \quad \left\| \frac{\varphi(z)}{z - \lambda_j} - \frac{\varphi(z)}{z - \lambda_k} \right\|_2 < \varepsilon.$$

Note that in this construction μ is a Clark measure for θ .

As in Grivaux' paper [8] we will proceed with the construction inductively. Namely, on the N -th step we construct $t_1, \dots, t_N \in [0, 1]$, μ_1, \dots, μ_N and functions θ_N defined by

$$(10) \quad i \frac{1 + \theta_N(z)}{1 - \theta_N(z)} := \sum_{n=1}^N \frac{\mu_n}{t_n - z},$$

and φ_N ,

$$(11) \quad \varphi_N(z) := (1 - \theta_N(z)) \left[1 + \sum_{n=1}^N \frac{c_n \mu_n}{t_n - z} \right],$$

where $c_n > 0$. The sequences μ_n and c_n will be assumed to tend to zero very rapidly.

It is a key idea of the construction that c_n are taken to be *positive*. In this case the function $1 + \sum_{n=1}^N \frac{c_n \mu_n}{t_n - z}$ (which appears in the definition of φ_N) is a Herglotz function (has positive real part in \mathbb{C}_+) and, therefore, it has exactly N zeros $\lambda_1^N, \lambda_2^N, \dots, \lambda_N^N$, interlacing with the points t_1, \dots, t_N . Choosing c_n sufficiently small we can control the location of these points.

Note that θ_N is a finite Blaschke product and the model space K_{θ_N} is N -dimensional. The measure $\sum_{n=1}^N \mu_n \delta_{t_n}$ is a Clark measure for K_{θ_N} . Also, $\varphi_N \notin K_{\theta_N}$, since $1 - \theta_N \notin K_{\theta_N}$, while

$$f_j^N(z) := \frac{\varphi_N(z)}{z - \lambda_j^N} \in K_{\theta_N}.$$

Indeed, we have a representation of type (5),

$$(12) \quad f_j^N(z) = \frac{\varphi_N(z) - \varphi_N(\lambda_j^N)}{z - \lambda_j^N} = (1 - \theta_N(z)) \sum_{n=1}^N \frac{c_n \mu_n}{(\lambda_j^N - t_n)(z - t_n)}.$$

Assume that t_n , μ_n and c_n , $1 \leq n \leq N_1$, are already chosen. On N -th step we will add a point $t_N \in (0, 1)$, its mass μ_N and a coefficient c_N in the following order. First, we take t_N to be very close to some zero of the function φ_{N-1} . Then we choose μ_N to be so small that θ_N does not differ much from θ_{N-1} outside a small neighborhood of t_N . Finally, we choose c_N to be even much smaller than μ_N so that all zeros $\lambda_1^N, \dots, \lambda_N^N$ from generation N almost coincide with the corresponding zeros $\lambda_1^{N-1}, \dots, \lambda_{N-1}^{N-1}$ from generation $N-1$, while the zero λ_N^N is very close to t_N .

Let us formally state what we need for the convergence.

(I) We will choose $\mu_n > 0$ and $c_n > 0$ so that $\mu_n < 2^{-n}$ and $c_n < 2^{-n}$. These conditions already ensure the convergence of the functions (10) and (11) to (7) and (8) respectively. Also, we will require that $|\lambda_j^{N-1} - \lambda_j^N| < 2^{-N}$, $j = 1, \dots, N-1$ (in fact we will need much more, see (18) below).

(II) Clearly, for each N , the functions f_j^N , $j = 1, \dots, N$, are linearly independent, and so they form a basis in K_{θ_N} . Let A_N be a sort of a ℓ^1 basis constant for $\{f_j^N\}$: for any $\{\alpha_j\}_{j=1}^N$, $\alpha_j \in \mathbb{C}$,

$$\sum_{j=1}^N |\alpha_j| \leq A_N \left\| \sum_{j=1}^N \alpha_j f_j^N \right\|_2.$$

Without loss of generality we assume that $A_N \geq 1$ and the sequence A_N increases. Our second requirement then reads as follows:

$$(13) \quad \|f_j^N - f_j^{N-1}\|_2 < \frac{1}{2^{N+2} A_{N-1}}, \quad j = 1, \dots, N-1.$$

(III) Let $l(n)$ be a sequence of integers such that $l(n) < n$ and $l(n)$ takes every integer value infinitely many times (e.g., $1, 1, 2, 1, 2, 3, 1, 2, 3, 4, \dots$). To achieve property (9), we will introduce the third requirement:

$$(14) \quad \|f_{l(N)}^N - f_N^N\|_2 < 2^{-N-1}, \quad N \in \mathbb{N}.$$

3.2. Choice of the parameters. Assume that t_n , μ_n and c_n , $n = 1, \dots, N-1$, are already chosen. First we choose the point t_N . Let ε_N be some small positive number (namely, let $\varepsilon_N \leq 4^{-N-2} A_{N-1}^{-1}$, where A_{N-1} is the constant from (II) above) and consider the equation

$$(15) \quad 1 + \sum_{n=1}^{N-1} \frac{c_n \mu_n}{t_n - x} = \varepsilon_N.$$

Clearly, it has $N-1$ real zeros which depend continuously on ε_N . Hence, they can be enumerated x_1, \dots, x_{N-1} in such a way that $x_j \rightarrow \lambda_j^{N-1}$ when $\varepsilon_N \rightarrow 0$.

Let us take ε_N to be so small that if we take as t_N the zero of (15) which is the closest to the point $\lambda_{l(N)}^{N-1}$, then

$$(16) \quad |t_N - \lambda_{l(N)}^{N-1}| < 4^{-3N} \delta_{N-1}^3,$$

where

$$\delta_{N-1} = \inf_{1 \leq j, n \leq N-1} |\lambda_j^{N-1} - t_n|.$$

Now we put $\mu_N = \sqrt{\varepsilon_N}$ and define the inner function θ_N by (10). One more additional restriction on the smallness of ε_N will be as follows:

$$(17) \quad \left\| \frac{\theta_N(z) - \theta_{N-1}(z)}{z - t_n} \right\|_2 < \frac{1}{4^N A_{N-1}}, \quad n = 1, \dots, N-1.$$

This is also possible by continuous dependence of these norms from ε_N . More precisely, by the construction of t_N we have $t_N \rightarrow \lambda_{l(N)}^{N-1}$ as $\varepsilon_N \rightarrow 0$, and so t_N is separated from t_n , $1 \leq n \leq N-1$. Therefore, we can choose a small neighborhood I of $\lambda_{l(N)}^{N-1}$ such that the integral of $\left| \frac{\theta_N(t) - \theta_{N-1}(t)}{t - t_n} \right|^2$ over I is small. Once the interval I is fixed, we can make the integral over $\mathbb{R} \setminus I$ to be arbitrarily small since $\theta_N - \theta_{N-1}$ (as well as $\theta'_N - \theta'_{N-1}$) tends to zero uniformly over $\mathbb{R} \setminus I$ as $\varepsilon_N \rightarrow 0$.

Finally, let us choose c_N . Consider the equation $\varphi_N(x) = 0$ which is equivalent to

$$1 + \sum_{n=1}^{N-1} \frac{c_n \mu_n}{t_n - x} + \frac{c_N \mu_N}{t_N - x} = 0.$$

Again, a continuity argument shows that we may enumerate the N zeros of this equation λ_j^N (which should be thought of as functions of c_N) so that $\lambda_j^N \rightarrow \lambda_j^{N-1}$, $j = 1, \dots, N-1$, and $\lambda_N^N \rightarrow t_N$ as $c_N \rightarrow 0$. Moreover, by the choice of t_N as a solution of (15), we have

$$1 + \sum_{n=1}^{N-1} \frac{c_n \mu_n}{t_n - \lambda_N^N} \rightarrow \varepsilon_N$$

and $\lambda_N^N \rightarrow t_N$ as $c_N \rightarrow 0$. Therefore,

$$\lambda_N^N - t_N \sim \frac{c_N \mu_N}{\varepsilon_N} = \frac{c_N}{\sqrt{\varepsilon_N}}, \quad c_N \rightarrow 0.$$

Let us now choose c_N to be so small that

$$(18) \quad |\lambda_j^N - \lambda_j^{N-1}| < \frac{1}{4^{3N} A_{N-1}} \delta_{N-1}^3, \quad j = 1, \dots, N-1,$$

$$(19) \quad \frac{c_N}{2\sqrt{\varepsilon_N}} \leq |\lambda_N^N - t_N| < |\lambda_{l(N)}^N - t_N|,$$

and

$$(20) \quad |\lambda_N^N - \lambda_{l(N)}^{N-1}| < 2^{-N} \delta_{N-1}^3.$$

The latter estimate is possible by (16) and (18).

3.3. Proof of (13). To estimate the norm $\|f_j^N - f_j^{N-1}\|_2$, $1 \leq j \leq N-1$, we write, using (12),

$$(21) \quad \begin{aligned} \frac{\varphi_{N-1}(z)}{z - \lambda_j^{N-1}} - \frac{\varphi_N(z)}{z - \lambda_j^N} &= \sum_{n=1}^{N-1} \frac{c_n \mu_n (1 - \theta_{N-1}(z))}{(t_n - \lambda_j^{N-1})(z - t_n)} \\ &\quad - \sum_{n=1}^{N-1} \frac{c_n \mu_n (1 - \theta_N)}{(t_n - \lambda_j^N)(z - t_n)} - \frac{c_N \mu_N (1 - \theta_N)}{(t_N - \lambda_j^N)(z - t_N)}. \end{aligned}$$

Denote the last summand in (21) by h . Then $h \in K_{\theta_N}$ (it is a reproducing kernel up to a coefficient), and so

$$\|h\|_2^2 = \frac{|c_N|^2 \mu_N}{|t_N - \lambda_j^N|^2} \leq 4\mu_N \varepsilon_N.$$

The first two summands in (21) may be rearranged into the sum of

$$g_1(z) = (1 - \theta_N(z)) \sum_{n=1}^{N-1} \left(\frac{c_n \mu_n}{(t_n - \lambda_j^{N-1})(z - t_n)} - \frac{c_n \mu_n}{(t_n - \lambda_j^N)(z - t_n)} \right)$$

and

$$g_2(z) = \sum_{n=1}^{N-1} c_n \mu_n \frac{\theta_N(z) - \theta_{N-1}(z)}{(t_n - \lambda_j^{N-1})(z - t_n)}.$$

We may compute the norm of $g_1 \in K_{\theta_N}$ using Clark's theorem (see formula (6)):

$$\|g_1\|_2^2 = \sum_{m=1}^{N-1} |g_1(t_m)|^2 \mu_m = \sum_{m=1}^{N-1} \frac{|\lambda_j^N - \lambda_j^{N-1}|^2 |c_m|^2 \mu_m}{|t_m - \lambda_j^{N-1}|^2 |t_m - \lambda_j^N|^2}.$$

Thus, $\|g_1\|_2$ does not exceed $4^{-N} A_{N-1}^{-1}$ by (18).

Finally, $\|g_2\|_2 \leq 4^{-N} A_{N-1}^{-1}$, and summing the above estimates we obtain (13).

3.4. Proof of (14). To estimate the norm $\|f_{l(N)}^N - f_N^N\|_2$ we use again Clark's theorem:

$$(22) \quad \left\| \frac{\varphi_N}{z - \lambda_{l(N)}^N} - \frac{\varphi_N}{z - \lambda_N^N} \right\|_2^2 = \sum_{m=1}^N \left| \frac{\varphi_N(t_m)}{t_m - \lambda_{l(N)}^N} - \frac{\varphi_N(t_m)}{t_m - \lambda_N^N} \right|^2 \mu_m.$$

Let $L_N = |\lambda_{l(N)}^N - \lambda_N^N|$. Then, by (18) and (20), $|t_m - \lambda_j^N| > L_N^{1/3}$ for any $m = 1 \dots N-1$ and $j = 1, \dots, N$. Now the first $N-1$ summands in (22) may be estimated as

$$\sum_{m=1}^{N-1} \frac{|\lambda_{l(N)}^N - \lambda_N^N|^2}{|t_m - \lambda_{l(N)}^N|^2 |t_m - \lambda_N^N|^2} |\varphi_N(t_m)|^2 \mu_m \leq \sum_{m=1}^{N-1} L_N^{2/3} |\varphi_N(t_m)|^2 \mu_m.$$

Since by (18) and (20) $L_N \leq 2^{-3N} \delta_{N-1}^3$ and $|\varphi_N(t_m)| = |\theta'_N(t_m)| c_m \mu_m = 2c_m$, we conclude that the latter sum does not exceed 2^{-N-1} .

It remains to consider the summand with the number N . We have

$$\left| \frac{\varphi_N(t_N)}{t_N - \lambda_N^N} \right| = \frac{2}{\mu_N} \cdot \frac{c_N \mu_N}{|t_N - \lambda_N^N|} \leq \frac{2\varepsilon_N}{\mu_N} = 2\sqrt{\varepsilon_N} \leq 2^{-N-2}.$$

Here we used inequality (19). The estimate for the term $\left| \frac{\varphi_N}{t_N - \lambda_{l(N)}^N} \right|$ is analogous. Estimate (14) is proved

3.5. Convergence and completeness. To ensure the pointwise convergences $\theta_N(z) \rightarrow \theta(z)$ and $\varphi_N(z) \rightarrow \varphi(z)$, $z \in \mathbb{C}_+$, it suffices to assume only that $\sum_n \mu_n < \infty$ and $c_n \rightarrow 0$.

By the choice of the parameters in Subsection 3.2, the sequence λ_j^N converges to some point λ_j , and it follows that f_j^N converges to $\frac{\varphi(z)}{z-\lambda_j}$ pointwise in \mathbb{C}_+ . On the other hand, by (13), the sequence $f_j^N \in H^2$ converges to some function f_j in H^2 (recall that $H^2 = H^2(\mathbb{C}_+)$ is the Hardy space in the upper half-plane). Since the convergence in H^2 implies the pointwise convergence in \mathbb{C}_+ , we conclude that $f_j(z) = \frac{\varphi(z)}{z-\lambda_j}$.

Next we prove that the family $\{f_j\}$ is complete in K_θ . First of all, note that if $g \in K_\theta$,

$$g(z) = (1 - \theta(z)) \sum_{n=1}^{\infty} \frac{d_n \mu_n}{z - t_n}, \quad g_N(z) = (1 - \theta_N(z)) \sum_{n=1}^N \frac{d_n \mu_n}{z - t_n},$$

then $\|g - g_N\|_2 \rightarrow 0$, $N \rightarrow \infty$. Indeed,

$$g(z) - g_N(z) = (\theta_N(z) - \theta(z)) \sum_{n=1}^N \frac{d_n \mu_n}{z - t_n} + (1 - \theta(z)) \sum_{n=N+1}^{\infty} \frac{d_n \mu_n}{z - t_n}.$$

The norm of the second sum by Clark's theorem (see (6)) equals $\sum_{m=N+1}^{\infty} |d_m|^2 \mu_m$, which obviously goes to zero, $N \rightarrow \infty$, while the norm of the first sum is small by the assumption (17).

Thus, we constructed a sequence $g_N \in K_{\theta_N}$ such that $g_N \rightarrow g$ in $L^2(\mathbb{R})$. It remains to approximate functions g_N by linear combinations of f_j . The method is borrowed from [8]. Since $\{f_j^N\}$ is a basis in K_{θ_N} , we may write $g_N = \sum_{j=1}^N \alpha_j f_j^N$. Then, making use of (13), we get

$$\begin{aligned} \|g - \sum_{j=1}^N \alpha_j f_j\|_2 &\leq \sum_{j=1}^N |\alpha_j| \sum_{k=N}^{\infty} \|f_j^k - f_j^{k+1}\|_2 \\ &\leq \frac{1}{2^N A_N} \sum_{j=1}^N |\alpha_j| \leq 2^{-N} \|g_N\|_2, \end{aligned}$$

which goes to 0 as $N \rightarrow \infty$. Completeness of the family $\{f_j\}$ is proved.

3.6. End of the proof of Theorem 1.3. Let us complete the proof of Theorem 1.3. We have constructed a complete sequence $\{f_j\} = \{\frac{\varphi}{z-\lambda_j}\}$ in K_θ . It remains to verify that $\{f_j\}$ has the property (9). Let $j \geq 1$ and ε be given. Choose N such that $l(N) = j$ and $2^{-N} < \varepsilon$, which is possible by the definition of the sequence $l(n)$. Then, by (14), $\|f_j^N - f_N^N\|_2 < 2^{-N-1}$. Also, by (14),

$$\|f_N^N - f_N\|_2 \leq \sum_{k=N}^{\infty} \|f_N^k - f_N^{k+1}\|_2 \leq 2^{-N-2}$$

and, analogously, $\|f_j^N - f_j\|_2 \leq 2^{-N-1}$. Combining these estimates we obtain $\|f_j - f_N\|_2 \leq 2^{-N}$. \square

4. CONCLUDING REMARKS

The unitary operator U in our construction (as well as in [8]) is of a very special form. Note that if $t_n \in \mathbb{R}$ is the sequence constructed in Section 3, then the spectrum of U is given by $\{\tau_n\}$, $\tau_n = \frac{t_n - i}{t_n + i}$. The points t_n are chosen inductively so that t_N be close to some of the points t_n , $1 \leq n \leq N - 1$, and so the set $\{t_n\}$ has a certain self-similarity. It seems to be a natural question, which unitary operators have hypercyclic rank one perturbations.

Problem 1. To describe cyclic unitary operators U such that $U + R$ is hypercyclic for some rank one operator R .

In particular, it is not clear whether the spectrum $\sigma(U)$ can have nonempty interior or positive measure.

Problem 2. Does there exist a cyclic unitary operator U such that $U + R$ is hypercyclic for some rank one operator R and $\sigma(U) = \mathbb{T}$.

As in [8], in the present paper the spectral measure of U is purely atomic.

Problem 3. Construct a unitary operator U , whose spectral measure is an absolutely continuous or a continuous (i.e., without point masses) singular measure on \mathbb{T} , such that $U + R$ is hypercyclic for some rank one operator R .

Note that the functional model applies to rank one perturbations of an arbitrary cyclic unitary operator whose spectral measure is singular. Therefore, one can hope to obtain further information about hypercyclic rank one perturbations of unitary operators using this model.

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